Quasi-invariant measures on orbits of the logic action

Logic Action

Countable relational $\mathcal{L}$

$\mathrm{Str}(\mathcal{L},\mathbb{N}) =$ "all $\mathcal{L}$-structures on domain $\mathbb{N}$"

A Polish space

$$= \prod_{R\in\mathcal{L}} \{0,1\}^{\overbrace{\mathbb{N}\times\cdots\times\mathbb{N}}^{\text{arity}(R)}}$$

$S_\infty = \{\text{all bijections } \mathbb{N} \longrightarrow \mathbb{N}\}$

A Polish group p.w. top

$S_\infty \curvearrowright \mathrm{Str}(\mathcal{L},\mathbb{N}) \qquad (g,\mathcal{M}) \longmapsto g_*\mathcal{M}$

the logic action

$$\boxed{g_* \mathcal{M} \models R(a_1, \dots, a_n) \text{ iff } \mathcal{M} \models R(g^{-1}(a_1), \dots, g(a_n))}$$

It is continuous $S_\infty \times Str(\mathcal{L}, \mathbb{N}) \to Str(\mathcal{L}, \mathbb{N})$

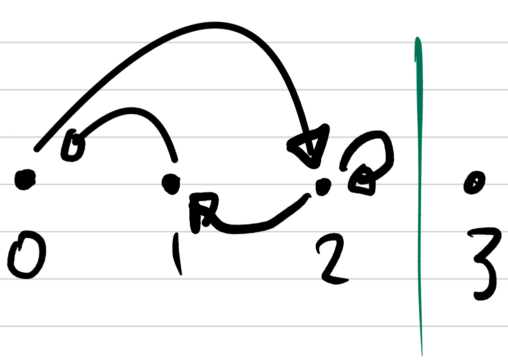
Orbit

$$S_\infty \cdot \mathcal{M} = [\mathcal{M}]_{iso} =: [\mathcal{M}]$$

Example $\mathcal{L} = \{R\}$ binary

$$\mathcal{M} \in Str(\mathcal{L}, \mathbb{N}) = \{0,1\}^{\mathbb{N} \times \mathbb{N}}$$

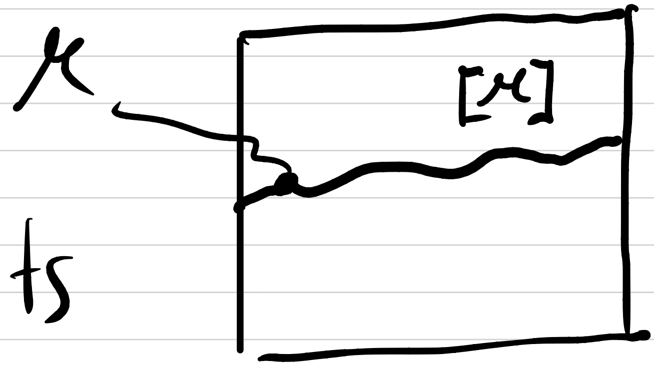
$$\mathcal{M} = \begin{array}{c|ccc|c} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 1 & \\ 1 & 1 & 0 & 0 & \\ 2 & 0 & 1 & 1 & \\ \hline 3 & & & & \end{array}$$



S_∞ act simultaneously on rows & columns

Large Orbits

$S_\infty \curvearrowright Str(\mathcal{L}, \mathbb{N})$



Question: which orbits are "large"

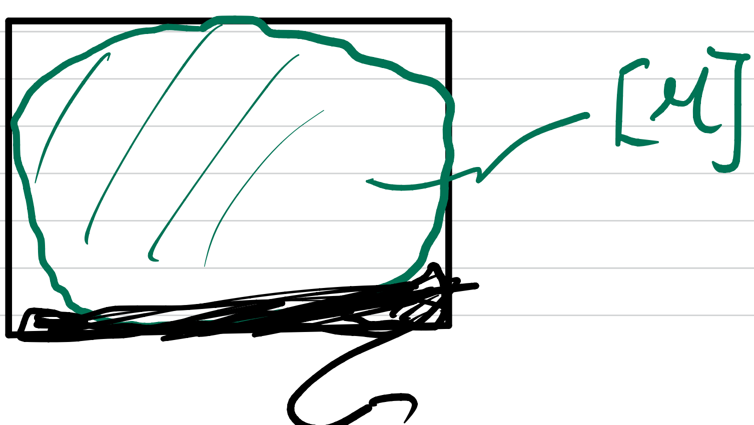
(w.r.t some (invariant) notion of largeness)

• Baire Category

Say $\mathcal{M}$ is "generic" is $[\mathcal{M}]$ comeager in $\overline{[\mathcal{M}]}$

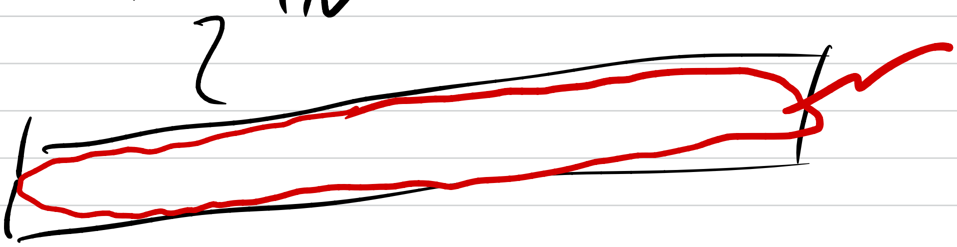
$\mathcal{L} = \{R\}$ in $Str(\mathcal{L}, \mathbb{N})$

we have a "generic" structure



$\mathcal{M}$ the Fraïssé limit of all finite Digraphs all graphs (symmetric irreflexive)

ZOOM IN



the Random graph

Is there a combinatorial way to characterize these $\mathcal{M}$

YES there is a "generic" structure inside $\overline{[\mathcal{M}]}$ iff $Age(\mathcal{M})$ has WAP

weak amalgamation property

<u>**Def**</u>: $Age(\mathcal{M})$ = all finite $\mathcal{A}$ with $\mathcal{A} \hookrightarrow \mathcal{M}$

$Age(\mathcal{M})$ has WAP

$$\forall A \; \exists \tilde{A} \; \forall \; \exists$$

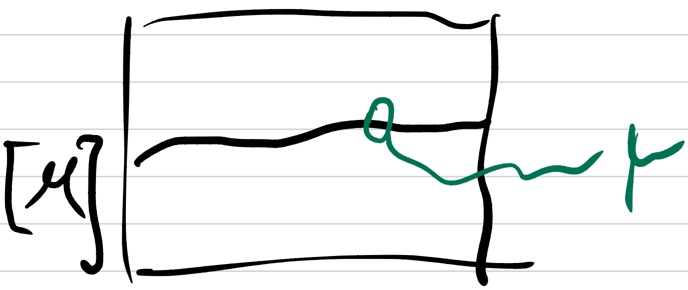
$$\begin{tikzcd} & & C \arrow[dr, "\bar{g}"] & \\ A \arrow[r, "i"] & \tilde{A} \arrow[ur, "g"] \arrow[dr, "f"'] & & D \\ & & B \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

$$\tilde{f} \circ f \circ i = \bar{g} \circ g \circ i$$

CAP

• Probability:

Say $\mathcal{M}$ is (invariantly) random if there is an S_∞-invariant probability measure μ on $[\mathcal{M}]$



Theorem (Ackerman–Freer–Patel)

$\boxed{\mathcal{M} \text{ is random}}$ iff $\mathcal{M}$ has no algebraicity

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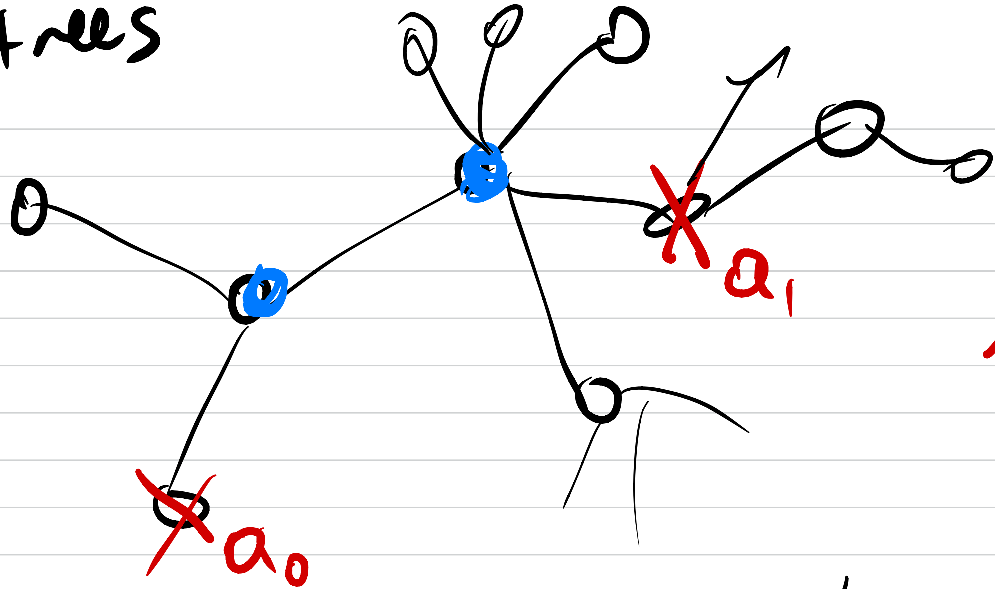
For any tuple $\bar{a}$ in M $\operatorname{acl}(\bar{a}) = \{\bar{a}\}$

where $\operatorname{acl}(\bar{a}) = \{b \in M : b \text{ has finite orbit under } \operatorname{Aut}(\mathcal{M})_{\bar{a}}\}$

trees



$\mathrm{Aut}(T)_{(a_0,a_1)}$

AFP Theorem II

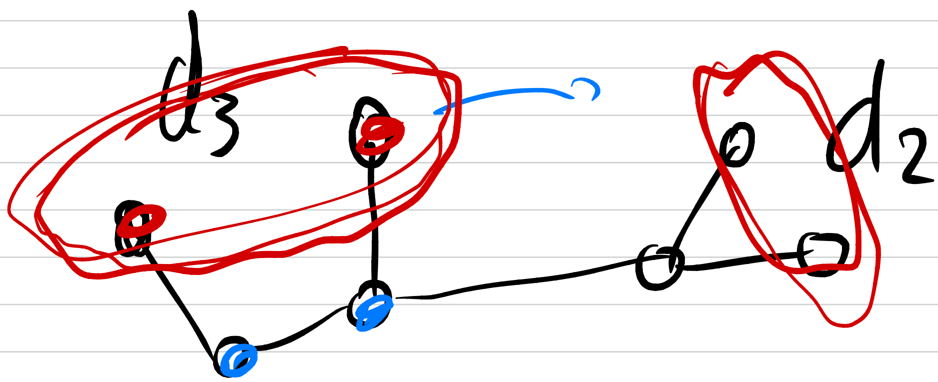
$\mathcal{M}$ is "random" iff $\mathcal{M}_{hom}$ is the Fraisse limit of $\mathrm{Age}(\mathcal{M}_{hom})$ and the latter has the strong amalgamation property

$\mathcal{M}$ is homogeneous iff

$\forall$ bijection
$$\begin{aligned} a_1 &\longrightarrow b_1 \\ a_2 &\longrightarrow b_2 \\ &\vdots \\ a_n &\longrightarrow b_n \end{aligned}$$

inducing isomorphism $\langle \bar{a} \rangle_{\mathcal{M}} \cong \langle \bar{b} \rangle_{\mathcal{M}}$

$\exists\, g \in \mathrm{Aut}(\mathcal{M}) \quad g(a_i) = b_i$



$T \rightsquigarrow T_{hom}$

add distances ?

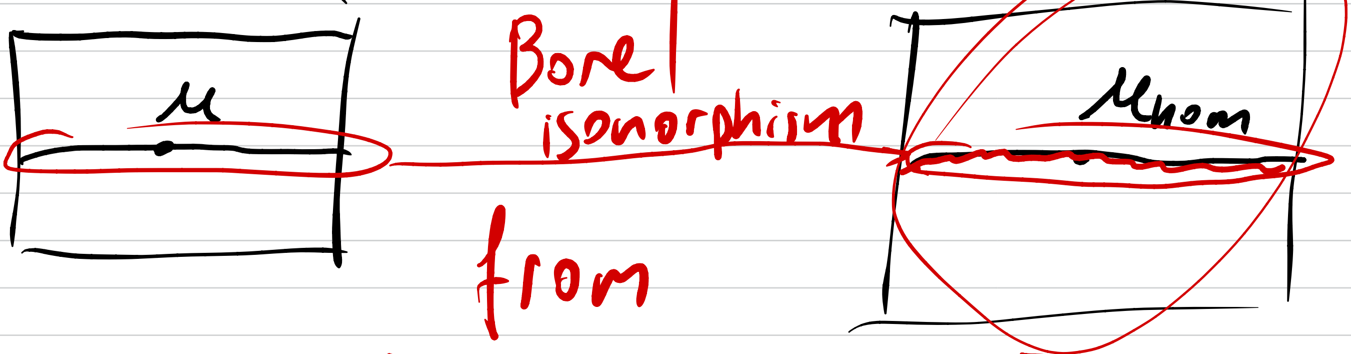
<u>Fact</u> By adding enough relations (∞-many in each arity) we can always find homogenization

$\mathcal{M} \rightsquigarrow \mathcal{M}_{hom}$

$\mathcal{M}_{hom}$ is homogeneous

$$Act(\mathcal{M}) = Act(\mathcal{M}_{hom})$$

$Str(L, \mathbb{N})$ $\quad S_\infty$-inv $\quad$ $Str(L_{hom}, \mathbb{N})$



Borel isomorphism from

$[\mathcal{M}]$ $\quad$ $[\mathcal{M}_{hom}]$

Def

$\mathcal{M}$ is quasi-random iff

there is an S_∞-quasi-invariant Borel probability μ on $[\mathcal{M}]$

$$\forall X \subseteq [\mathcal{M}] \quad \forall g \in S_\infty$$

$$\mu(X) = 0 \iff \mu(gX) = 0$$

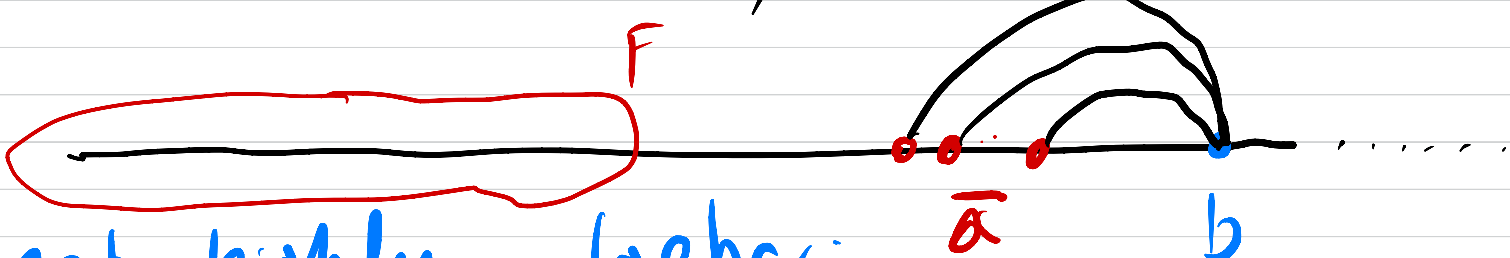
Theorem (Conley, Jahel, P.)

$\mathcal{M}$ is quasi-random iff $\mathcal{M}$ is not highly algebraic

Def

$\mathcal{M}$ is highly algebraic if for every finite $F \subseteq M$ there are $\bar{a}$, b with $\{\bar{a}\}, \{b\}, F$ pair-wise disjoint

s.t. $b \in acl(\bar{a})$



$a_i \in acl(\bar{a})$

not highly algebraic

$\exists$ F

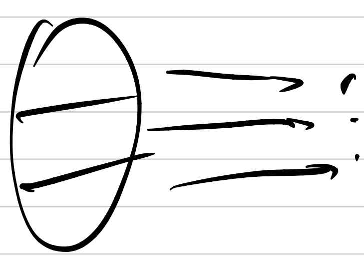
$b \in F$ can be in $acl(\bar{a})$

Examples of μ with no high alg.

i.e. quasi-random

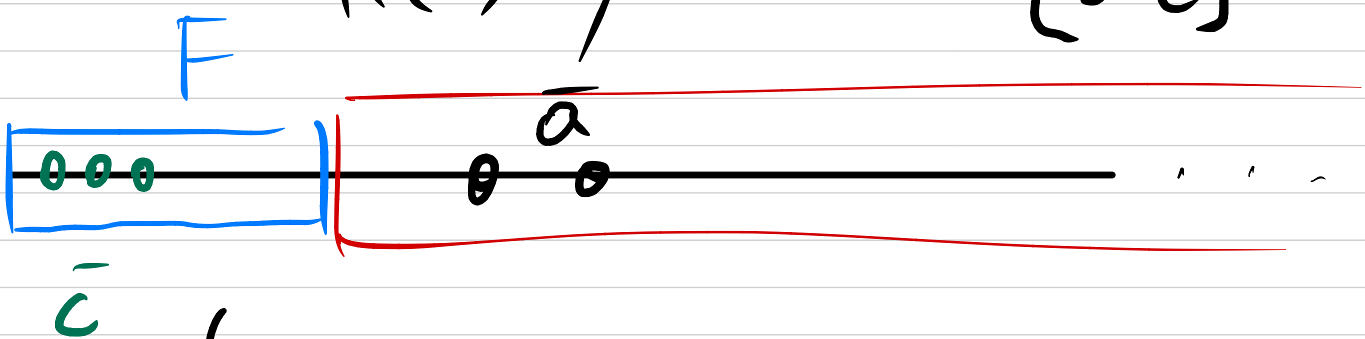
(1) Pick any μ without algebr.
name finitely many points

(2) Pick any μ without algebr
Pick finite set F
Pick generic surj $M \to F$

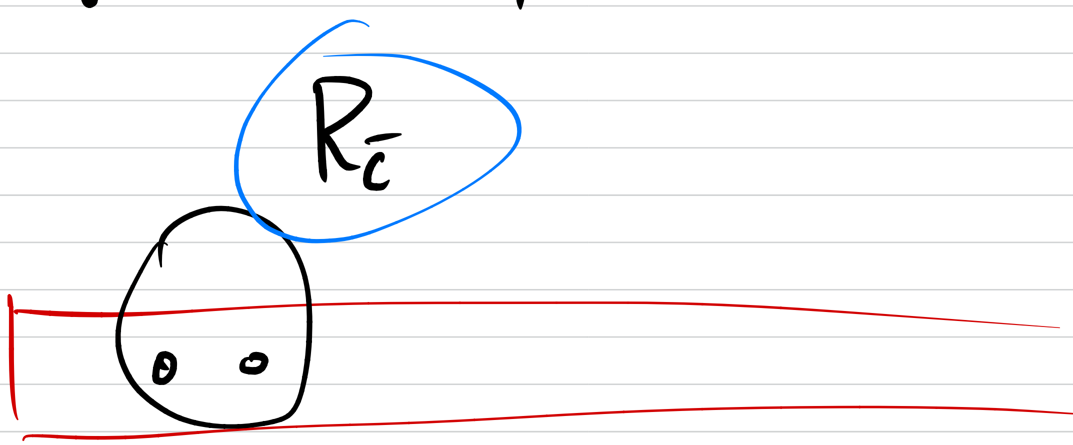


Comments on the proof

• If $\mathcal{M}$ is not highly algebraic want to build quasi-inv f $R(\bar{c}, \bar{a})$ on $[\mathcal{M}]$



$\Downarrow$ define $\mathcal{M}_F$ in $\mathcal{L}_F$



$\forall \bar{c}$ arity m in F

$\forall R$ arity $n \geq m$

$\mathcal{M}_F$ has no algebraicity

AFP

invariant f_F on $[\mathcal{M}_F]$

Combine it with ν on the countable set $\mathbb{N}^F$

fully supported

0^F

If $\mathcal{M}$ is highly algebraic then no quasi-inv on $[\mathcal{M}]$

How to show that no quasi-inv measure exists?

$g_s \quad s \in 2^{<\mathbb{N}}$

Instead show:

for every compact $K \subseteq [\mathcal{M}_{hom}]$ there is $(g_\alpha : \alpha \in 2^{\mathbb{N}})$ in S_∞

s.t. $\alpha \neq \beta$ $\quad g_\alpha K \cap g_\beta K = \emptyset$

Suffices case

Regularity of Prob measures

$\exists K \subseteq [M]$ $\quad \mu(K) > 0$

Quasi-inv $\quad \mu(g_\alpha K) > 0$ for all α

$\exists \varepsilon > 0$ $\quad \{\alpha \in 2^{\mathbb{N}} : \mu(g_\alpha K) > \varepsilon\}$ uncountable

Contradiction by σ-additive

Naive Hope

$\mu(X) = 0$

$\mu(gX) \neq 1$

M has locally finite algebraicity

acl is locally finite

$\forall$ finite $a_1, \dots, a_n$

$\mathrm{acl}(\bar{a})$ is finite

$\Updownarrow$

admitting Quasi-invariant measure

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e.g. $G = S_\infty$

Also Tsankov (2025)

Theorem If $G \leqslant S_\infty$ Roelcke pre-compact

& $G \curvearrowright (X, \mu)$ quasi-invariant

then $X = \bigsqcup_n X_n$ s.t. each X_n G-invariant

& $\mu \upharpoonright X_n \simeq_{iso}$ induced $\left(\begin{array}{l} H \curvearrowright (X_n, \nu_n) \\ \text{p.m.p} \\ H \leqslant G \text{ open} \end{array} \right)$